NORTH-HOLLAND

A Discrete Inverse Vibration Problem with Parameter Uncertainty

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ABSTRACT

Inverse vibration problems consider the use of eigenvalue data to calculate the properties of a linear dynamic system. The present study examines the effects of uncertainty in such problems. The type of problem studied is limited to simple spring-mass systems, that is, undamped discrete systems. A procedure is available for finding the values of these spring constants and masses from two sets of eigenvalues, one from the system with one free end, and one from the same system with a more restrictive boundary condition. Expressions are derived for estimating the statistics of these system parameters using the statistics of the eigenvalues. These expressions are based on first-order and second-order perturbation expansions and require the evaluation of partial derivatives of the stiffnesses and masses in terms of the eigenvalues. Results are obtained for several systems which differ in amount of uncertainty, boundary conditions, and number of degrees of freedom. These results are compared to those given by Monte Carlo simulation. The results show excellent agreement for sufficiently small degrees of randomness, as would be expected from using a perturbation method.

APPLIED MATHEMATICS AND COMPUTATION 69:313–333 (1995)

655 Avenue of the Americas, New York, NY 10010

0096-3003/95/$9.50
SSDI 0096-3003(94)00140-Y

1. INTRODUCTION

The past few decades have seen an increase in the importance of probabilistic methods in structural dynamics. One topic of particular interest is that of *random eigenvalues* [1].

The eigenvalues and eigenvectors of a linear dynamic system provide important information about its characteristics and behavior. These parameters are determined by the system's mass, stiffness, and damping properties, as well as its boundary conditions and structural flaws. In the mathematical formulation of a physical system, these properties are usually only approximately known. It may be appropriate, therefore, to model them as random variables. As functions of these properties, the eigenvalues and eigenvectors are themselves random, and knowledge of their distributions is of great importance in understanding the system's behavior.

The goal of this study is the application of probabilistic modeling to particular *inverse* problems in vibration. In these inverse problems, some of the eigenvalues and (possibly) eigenvectors are known quantities, from which the physical system properties are to be obtained. From a probabilistic point of view, the eigenvalues are considered to be random quantities, presumably due to experimental error in their measurement. The system property statistics can then be determined. In this way, a model can be examined in a more realistic manner. In addition, such an approach allows for estimation of the confidence level in the computed output. One can quantify, for example, the probability that the value of a particular output parameter will fall within a specified range.

Possible applications of this work include a method for nondestructive testing/evaluation (NDT/E) of structures. Periodic measurements would be able to detect shifts in the spectral properties of a given structure over time. In particular, such techniques could be utilized to estimate and locate changes in structural stiffness due to aging.

Within such a program, there are several factors which make a probabilistic approach desirable. First, it is necessary to detect very small changes in the structure's stiffness characteristics so that appropriate action can be taken before damage is too severe. Consequently, the magnitude of the uncertainty may be significant when compared with the magnitude of these small changes.

Also, it is necessary to relate changes in stiffness parameters to the magnitude and location of possible damage. Such relationships are likely to be somewhat uncertain in nature. In this case, it would be inconsistent to treat the stiffness changes as deterministic values.

A discussion of some (deterministic) inverse vibration problems is given

by Gladwell [2, 3]. These sources also provide a historical review of such problems.

Several types of problems in probabilistic structural dynamics are reviewed by Ibrahim [4]. This paper discusses methods employed in the analysis of random eigenvectors and random response characteristics. Included in this discussion is an outline of a perturbation technique to be used in random eigenvalue problems.

Collins and Thomson [5] provide a more detailed look at the use of perturbation methods in random eigenvalue problems. Numerical results are given in addition to the theoretical background. These results are shown to compare favorably with those obtained from Monte Carlo analysis.

In the application of perturbation techniques, sensitivity analysis is of great interest. Fox and Kapoor [6], Rogers [7], and Rudisill [8] present methods to calculate the first derivatives of eigenvalues and eigenvectors with respect to masses and stiffnesses in a discrete linear system. Brandon [9] examines cases in which this type of first-order sensitivity analysis is inadequate, and second-order approximations provide more acceptable results.

As mentioned earlier, the objective of this study is the application of probabilistic methods in the analysis of inverse problems in vibration. In particular, a method is presented by which the statistics of a linear discrete system's masses and spring constants can be estimated using statistical eigenvalue data. A perturbation expansion is used to derive expressions for the system statistics in terms of partial derivatives of the system parameters. Where necessary, it has been assumed that the random eigenvalue distributions satisfy certain conditions, such as uniform probability density functions and statistical independence.

Results are given for several example cases. A two-degree-of-freedom system is analyzed, and results are compared with those obtained using Monte Carlo simulation. Agreement between these results indicates accuracy of the perturbation method when randomness of the eigenvalues is sufficiently small.

2. THE DETERMINISTIC INVERSE VIBRATION PROBLEM

In a typical vibration problem, the physical parameters of the system are (at least approximately) known. These parameters take the form of masses and spring constants for a discrete system, or density, modulus of elasticity, and physical dimensions for a continuous system. From an analysis of these parameters, the natural frequencies or the response to a particular excitation can be determined.

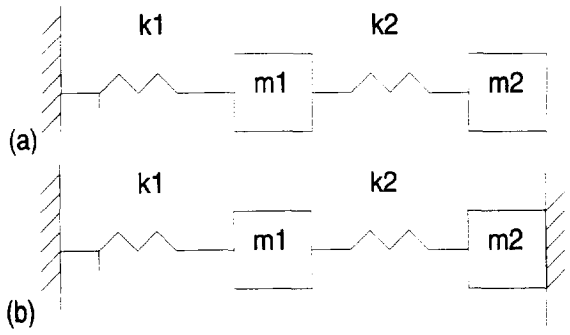

FIG. 1. A two-degree-of-freedom system.

In an *inverse* vibration problem, the physical parameters of a system are determined from the spectral data (i.e., frequencies and mode shapes, or eigenvalues and eigenvectors). More details are given by Gladwell [2, 3].

Consider the simple spring-mass system shown in Figure 1a. From vibration theory, it is known that this system has two distinct positive eigenvalues, λ_1 and λ_2, which are the roots of the characteristic equation

$$\lambda^2 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right]\lambda + \frac{k_1 k_2}{m_1 m_2} = 0. \tag{1}$$

This equation yields the following relations between the eigenvalues:

$$\lambda_1 + \lambda_2 = \frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}, \qquad \lambda_1 \lambda_2 = \frac{k_1 k_2}{m_1 m_2}. \tag{2}$$

In inverse vibration problems, the goal is to use the eigenvalue data to reconstruct the physical system properties. For this example case, there are two equations for the four unknown values k_1, k_2, m_1, and m_2. This implies that there are an infinite number of two-degree-of-freedom models which have the eigenvalues λ_1 and λ_2. Thus, it is necessary to introduce two more equations so that the system can be specified.

In order to obtain more equations, consider the system shown in Figure 1b. This system is identical to the previous one, except that the right end has been fixed, restricting it to a single degree of freedom. This constrained system has a single eigenvalue λ_3 which is given by

$$\lambda_3 = \frac{k_1 + k_2}{m_1}. \tag{3}$$

Again, it will be assumed that this eigenvalue is a known quantity, so that constraining an end is one way of obtaining an additional equation.

By manipulating (2) and (3), the ratios between the system properties can be obtained as

$$\begin{aligned} R_1 &\equiv \frac{k_2}{m_2} = \lambda_1 + \lambda_2 - \lambda_3 \\ R_2 &\equiv \frac{k_1}{m_1} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2 - \lambda_3} \\ R_3 &\equiv \frac{k_2}{m_1} = \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}{\lambda_1 + \lambda_2 - \lambda_3}. \end{aligned} \tag{4}$$

Ratios R_1, R_2, and R_3 must all be positive if there is to be a corresponding physical system. This requires that the eigenvalues satisfy

$$0 < \lambda_1 < \lambda_3 < \lambda_2,$$

which is predicted for this type of system by the inclusion principle [10].

These ratios obviously reveal a great deal about the dynamic properties of the system, but they do not uniquely identify it. In order to do this, some further information is needed. For example, if the total mass of the system, $m = m_1 + m_2$, is known, then the parameters can be uniquely solved in terms of (4),

$$\begin{aligned} k_1 &= \frac{R_1 R_3}{R_1 + R_3} m \\ k_2 &= \frac{R_1 R_2}{R_1 + R_3} m \\ m_1 &= \frac{R_1}{R_1 + R_3} m \\ m_2 &= \frac{R_3}{R_1 + R_3} m. \end{aligned} \tag{5}$$

The system's total mass serves only to scale the results, so in many cases it may be sufficient to assume a value if it is not known explicitly.

It has been shown that it is possible to derive closed-form solutions to the inverse vibration problem associated with a two-degree-of-freedom discrete system. This is possible, but impractical, for larger systems, such as that shown in Figure 2. Here, the two systems are identical except for the extra spring, k_{N+1}, at the right end of system (b). Note that the example from the previous section is easily seen to be a special case of this with $N = 2$ and $k_3 \to \infty$. The procedure described by Gladwell [2] is

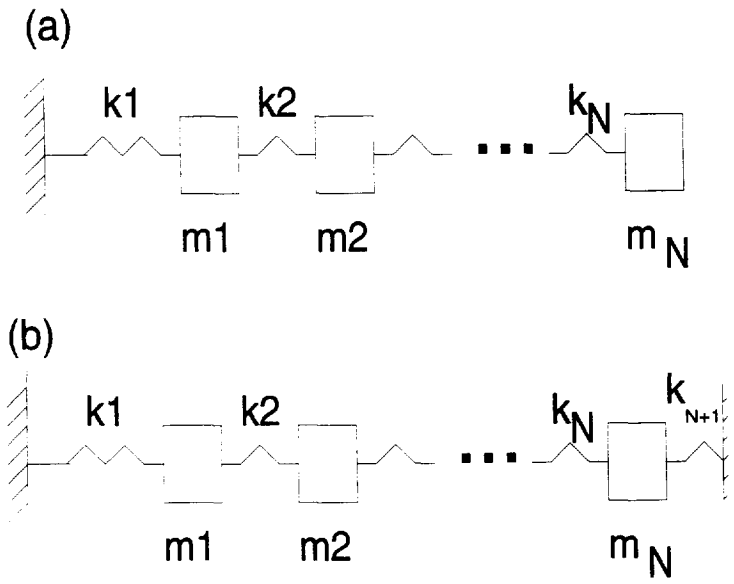

FIG. 2. An N-degree-of-freedom system.

more useful in dealing with such systems. The iterative procedure is stable and quite straightforward, indicating that it can be easily programed on a computer.

3. PROBABILISTIC PERTURBATION METHOD

Up to this point, it has been assumed that all quantities are deterministic. However, quantities involved in the design and analysis of engineering systems generally exhibit some degree of randomness. This can be attributed to several sources. One such source is the uncertainty involved in measurements. A measurement may be made, in some cases, as precisely as is needed for a particular application. In other situations, a measurement can be made only as precisely as the technology of the measuring system will allow. In either case, there is some uncertainty in the resulting values.

Even if quantities could be exactly measured, the inherent statistical nature of material properties and production techniques suggests a need for probabilistic methods. Two seemingly identical components will, in general, exhibit slight differences which may affect their respective performances. Assemblies of such components are even more likely to differ from one to another.

Finally, the modeling of engineering systems usually requires some approximation. Reasons for this include a lack of understanding of the particular system or a need to simplify a particularly complex equation. Such assumptions may introduce some form of uncertainty into the solution. This last form of randomness obviously depends on the particular system. The analysis of this type of uncertainty can be quite difficult.

This analysis considers only randomness of the first two types. This means that the system can be mathematically modeled using deterministic equations, while randomness is introduced in the variables. It is important to note that a probabilistic analysis of the type carried out in this work not only provides a better model of the system, but also provides the analyst with a tool for quantifying statistical confidence in the analytical results.

A continuous random variable x can be completely characterized by its *probability density function* $f(x)$. The density function is defined such that

$$P(a < x < b) = \int_a^b f(x)\,dx, \tag{6}$$

where $P(a < x < b)$ represents the probability that the value of x lies between the real constants a and b. In practice, it is more useful to use statistics such as the mean value and standard deviation to quantitatively summarize the random properties of the variable. The *mean value*, or expected value, of variable x is given by the expression

$$E[x] = \int_{-\infty}^{\infty} x f(x)\,dx, \tag{7}$$

while the standard deviation is the square root of the variance, defined by

$$\begin{aligned} \sigma_x^2 &= \int_{-\infty}^{\infty} (x - E[x])^2 f(x)\,dx \\ &= E[x^2] - E^2[x]. \end{aligned} \tag{8}$$

The mean value gives a measure of the central tendency of the random variable. It can be thought of as an "average" value, while the standard deviation measures the amount of scatter. This can be normalized to give the coefficient of variation, defined by

$$\mathrm{COV}_x = \frac{\sigma_x}{E[x]}. \tag{9}$$

The definitions above can be extended to calculate the statistics of an

arbitrary function $F(x)$ as

$$E[F(x)] = \int_{-\infty}^{\infty} F(x) f(x)\, dx \tag{10}$$

and

$$\sigma^2_{F(x)} = E[F(x)^2] - E^2[F(x)]. \tag{11}$$

Consider a function F of random variables $\lambda_i (i = 1, 2, \ldots, n)$. Each of these variables can be written as

$$\lambda_i = \overline{\lambda_i} + \epsilon_i, \tag{12}$$

where $\overline{\lambda_i}$ is the mean value of λ_i and ϵ_i is a (small) random parameter. Note that this implies that $E[\epsilon_i] = 0$ and $E[\epsilon_i^2] = \sigma^2_{\lambda_i}$. The function F can now be expanded into a Taylor series with respect to the random variables ϵ_i as

$$F = \overline{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \epsilon_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 F}{\partial \lambda_i \, \partial \lambda_j} \epsilon_i \epsilon_j + \cdots. \tag{13}$$

In this expression, $\overline{F}$ is not the mean value of F, but the value of the function calculated at the mean values of the λ_i's. The derivatives are likewise calculated at the mean values. If the variables exhibit only a small degree of randomness, that is, if each ϵ_i is small, the expansion can be truncated after only a few terms with little error.

Taking the first two terms of (13) gives a linear approximation for the actual value of F:

$$F = \overline{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \epsilon_i. \tag{14}$$

Since $E[\epsilon_i] = 0$, taking the expected value of F leaves

$$E[F] = \overline{F}. \tag{15}$$

The analysis can be simplified by assuming the variables to be statistically independent, so that $E[\epsilon_i \epsilon_j] = E[\epsilon_i] E[\epsilon_j] = 0$ for $i \neq j$. The standard deviation of F is thus estimated by the difference

$$\sigma_F^2 = E[F^2] - E^2[F] = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \lambda_i} \right)^2 \sigma^2_{\lambda_i}. \tag{16}$$

It should be noted that (16) depends only on the means and standard deviations of the random variables. It is independent of the particular

distribution of these variables, with the only assumption being that they are independent. The method can therefore prove to be useful in cases where little is known about the nature of the random variables, but estimates of their statistics can be obtained.

A somewhat more accurate prediction for the statistics of random variable F would be expected by retaining the second-order term of the Taylor series,

$$F = \overline{F} + \sum_{i=1}^{n} \frac{\partial F}{\partial \lambda_i} \epsilon_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 F}{\partial \lambda_i \, \partial \lambda_j} \epsilon_i \epsilon_j. \tag{17}$$

In calculating expected values for this expansion, several assumptions will simplify the expressions:

(1) As before, the variables λ_i are assumed to be independent. This means that the expected value of a product of different variables can be written as the product of their respective mean values. This further implies that $E[\epsilon_i^p \epsilon_j^q \cdots] = 0$ if one of the exponents is equal to one. For example, $E[\epsilon_1^2 \epsilon_2] = E[\epsilon_1^2] E[\epsilon_2] = 0$.

(2) The probability density function of each λ_i is symmetric about the mean. This implies that the $E[\epsilon_i^n] = 0$ when n is odd.

(3) The probability density function of each λ_i is completely determined by two parameters, the mean and standard deviation. Since the mean for ϵ_i is zero, this allows for calculation of $E[\epsilon_i^4] = \beta_2 E^2[\epsilon_i^2]$ where the value of β_2 depends upon the type of distribution (i.e., uniform, normal, etc.).

One distribution which satisfies these criteria is the uniform distribution. By definition, such distributions are independent, and they are obviously symmetric about the mean values. Furthermore, for a uniform distribution, $\beta_2 = \frac{9}{5}$. The results given later are based on the assumption of uniform distributions for the eigenvalues.

In light of these assumptions, taking the expected value of F now gives

$$E[F] = \overline{F} + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 F}{\partial \lambda_i^2} E[\epsilon_i^2], \tag{18}$$

and the desired standard deviation can be calculated as

$$\begin{aligned} \sigma_F^2 \;=\; & \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \lambda_i} \right)^2 \sigma_{\lambda_i}^2 + \frac{1}{4} (\beta_2 - 1) \sum_{i=1}^{n} \left(\frac{\partial^2 F}{\partial \lambda_i^2} \right)^2 \sigma_{\lambda_i}^4 \\ & + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} {}' \left(\frac{\partial^2 F}{\partial \lambda_i \, \partial \lambda_j} \right)^2 \sigma_{\lambda_i}^2 \sigma_{\lambda_j}^2. \end{aligned} \tag{19}$$

The prime following the double summation symbols in this expression have been used to indicate that the sum is to include only terms where $i \neq j$. This is necessary since $i = j$ is already accounted for in the ϵ_i^4 summation.

The first term of this expression is simply the result for a first-order expansion. The remaining terms consist of sums of squared values. Therefore, if the assumptions discussed earlier are held to be true and $\beta_2 > 1$, then the second-order expansion yields higher standard deviations than the first-order expansion. This may be interpreted to mean that the second-order expansion will result in a somewhat more conservative estimate of the standard deviation than will the first-order expansion.

It can be shown that a third-order expansion gives the same expression as the second-order expansion, with some additional terms. In this case, however, the summations contain mixed products of derivatives rather than the squares of derivatives. This implies that some of the terms may be negative, so that no general statement can be made for the magnitude of the third-order results as compared with the first- and second-order results.

It is necessary to verify solutions obtained using these approximate expansions. One powerful way is through the utilization of Monte Carlo simulation techniques. The method is not discussed here since it is well documented and its theory is not the primary interest of this work.

4. CASE STUDIES

The previously discussed inverse vibration problem essentially reduces to the calculation of each mass and spring constant as a function of two sets of eigenvalues, one from the unconstrained system and the other from the constrained system. These functions are, in general, difficult to obtain in closed form. This difficulty is compounded by the need to evaluate partial derivatives in the implementation of the probabilistic perturbation method. For this reason, a FORTRAN program has been written to calculate these derivatives numerically, and use them to compute the approximate statistics of the system parameters.

Rather than choosing sets of eigenvalues arbitrarily, the examples in the following sections were selected by finding the eigenvalues of particular mass-spring systems. These eigenvalues are then used as mean values in the subsequent analyses.

4.1. A two-degree-of-freedom system

A two-degree-of-freedom system was selected with

$$k_1 = k_2 = k_3 = m_1 = m_2 = 10.$$

Note that only the relative effects of uncertainty are of interest in this study, so that the specification of units is not required. The corresponding eigenvalues can be easily found from classical vibration theory to be

$$\begin{aligned} \lambda_1 &= 0.38197, \\ \lambda_2 &= 2.61803 \end{aligned}$$

for the unconstrained system, and

$$\begin{aligned} \lambda_1^{\bullet} &= 1.00000, \\ \lambda_2^{\bullet} &= 3.00000 \end{aligned}$$

for the system when constrained by the additional spring k_3. Using these as mean values, assuming a standard deviation for each, and assuming a value for the total system mass, m (20 in this case) gives enough information to estimate the statistics of the springs and masses.

It is assumed that all eigenvalues within an example case have the same coefficient of variation, denoted by COV_λ. Thus, the higher eigenvalues will have more uncertainty than will the lower ones. For a first run, it was assumed that $\mathrm{COV}_\lambda = 0.01$, while the total mass is deterministic (i.e., $\sigma_m = 0$).

The Monte Carlo method was used in order to provide a basis for comparison. The number of iterations used was 5000, since little change was observed in the estimated statistics when more iterations were used.

Table 1 shows the results produced by Monte Carlo simulation, followed by those found using the perturbation method. The table shows excellent agreement between the three sets of results. Assuming the Monte Carlo results to be "exact," the second-order approximation produces results which are slightly more accurate than the first-order approximation, as expected.

Note that the standard deviations of the two masses are identical within each set of results. This is a consequence of the fact that the sum of the two mass values must be the total system mass, which is deterministic. From a numerical point of view, this implies that the derivatives of m_1 must be equal in magnitude, though opposite in sign, to the derivatives of m_2. The variables m_1 and m_2 are thus negatively correlated.

The accuracy of the perturbation method depends, of course, on the degree of randomness of the variables, as given by the magnitude of their standard deviations. Indeed, an increase in COV_λ from 0.01 to 0.03 shows a marked increase in the difference between the Monte Carlo and perturbation results. These results, shown in Table 2, indicate good agreement

TABLE 1
RESULTS FOR COV_λ OF 0.01

Method	Parameter	Results			Percent difference from Monte Carlo	
		Mean value	Standard deviation	Coefficient of variation	Mean value	Standard deviation
Monte Carlo simulation	m_1	9.9933	0.23579	0.02359	—	—
	m_2	10.007	0.23579	0.02356	—	—
	k_1	10.021	0.21737	0.02169	—	—
	k_2	9.9781	0.16299	0.01634	—	—
	k_3	9.9937	0.25554	0.02557	—	—
First-order perturbation expansion	m_1	10.000	0.23452	0.02345	0.07	0.54
	m_2	10.000	0.23452	0.02345	0.07	0.54
	k_1	10.000	0.21218	0.02122	0.21	2.45
	k_2	10.000	0.15816	0.01582	0.22	3.05
	k_3	10.000	0.25496	0.02550	0.06	0.23
Second-order perturbation expansion	m_1	9.9950	0.23466	0.02348	0.02	0.48
	m_2	10.005	0.23466	0.02346	0.02	0.48
	k_1	10.019	0.21334	0.02129	0.02	1.89
	k_2	9.9780	0.16027	0.01606	0.00	1.70
	k_3	9.9970	0.25501	0.02551	0.03	0.21

for the calculated mean values, but large differences can be seen in the standard deviations of the spring constants k_1 and k_2. The second-order expansion produces a better prediction for σ_{k_2} than does the first-order expansion, but this calculated value is still nearly 20% lower than that given by the Monte Carlo simulation. In the other parameters, the differences between the first- and second-order approximations are negligible.

A different view of the effects of eigenvalue uncertainty on the results is given in Figures 3–5. These figures show the coefficient of variation of some of the system parameters as a function of COV_λ. For comparison, curves are shown for first- and second-order approximations, as well as Monte Carlo results. Similar plots can be produced for the remaining parameters.

A limitation to the application of the Monte Carlo method to this problem should be noted here. Recall that the eigenvalues must satisfy the inequality

$$0 < \lambda_1 < \lambda_1^\bullet < \lambda_2 < \lambda_2^\bullet < \cdots < \lambda_N < \lambda_N^\bullet.$$

If this condition is violated, the given algorithm will not work. There is a chance that this will occur during the Monte Carlo simulation if there

TABLE 2
RESULTS FOR COV_λ OF 0.03

Method	Parameter	Results			Percent difference from Monte Carlo	
		Mean value	Standard deviation	Coefficient of variation	Mean value	Standard deviation
	m_1	9.9620	0.68698	0.06896	—	—
Monte	m_2	10.038	0.68698	0.06844	—	—
Carlo	k_1	10.233	0.87597	0.08560	—	—
simulation	k_2	9.7780	0.63331	0.06477	—	—
	k_3	9.9523	0.79002	0.07938	—	—
	m_1	10.000	0.70355	0.07036	0.38	2.36
First-order	m_2	10.000	0.70355	0.07036	0.38	2.36
perturbation	k_1	10.000	0.63655	0.06366	2.33	37.61
expansion	k_2	10.000	0.47447	0.04745	2.22	33.48
	k_3	10.000	0.76488	0.07649	0.48	3.29
	m_1	9.9550	0.70745	0.07107	0.07	2.89
Second-order	m_2	10.045	0.70745	0.07043	0.07	2.89
perturbation	k_1	10.171	0.66723	0.06560	0.61	31.28
expansion	k_2	9.8020	0.52888	0.05396	0.24	19.75
	k_3	9.9730	0.76622	0.07683	0.21	3.11

is any overlap of the probability density functions of the eigenvalues. Assuming that all eigenvalues have the same coefficient of variation and the distributions are uniform, the condition

$$\lambda_i < \lambda_j$$

will be guaranteed if

$$\text{COV}_\lambda < \frac{1}{\sqrt{3}} \frac{(\lambda_j - \lambda_1)}{(\lambda_i + \lambda_j)},$$

which, for this case, gives the maximum $\text{COV}_\lambda = 0.03925$. This explains the limited range of the Monte Carlo result curves.

Note that, as predicted, the second-order approximations are greater than or equal to the first-order ones. For the parameters m_1 and m_2, however, the Monte Carlo simulation gives results which are actually less than the first-order results, so that the perturbation method gives conservative estimates.

In contrast, the coefficients of variation of the spring constants are pre-

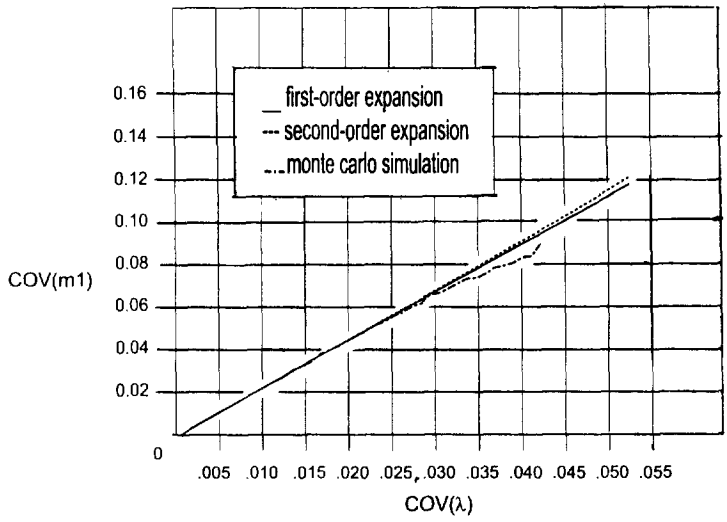

FIG. 3. Effect of COV_λ on COV_{m_1}, $k_3 = 10$.

dicted by both perturbation methods to the lower than those given by Monte Carlo simulation. This means that the second-order expansion gives more accurate results, although the difference in accuracy is small.

4.2. Effect of boundary conditions

In order to determine the system parameters for an N-degree-of-freedom system using the previous analysis, it is necessary to use eigenvalue data for the system with a second boundary condition. This is modeled as a constraint in the form of an additional spring with constant k_{N+1}. Thus, this constraint spring can be thought of as an "artificial" parameter, which is actually external to the system of interest, and is used only for the purpose of reconstructing that system.

For this reason, it is possible to consider the spring k_3 to be a parameter which can be, to some degree, controlled within an experimental measurement process. It may be of interest, therefore, to look at the results obtained by changing the value of k_3 from the previous example. Increasing it by an order of magnitude from 10 to 100 results in eigenvalues given by

$$\lambda_1^\bullet = 1.89023$$
$$\lambda_2^\bullet = 11.10977$$

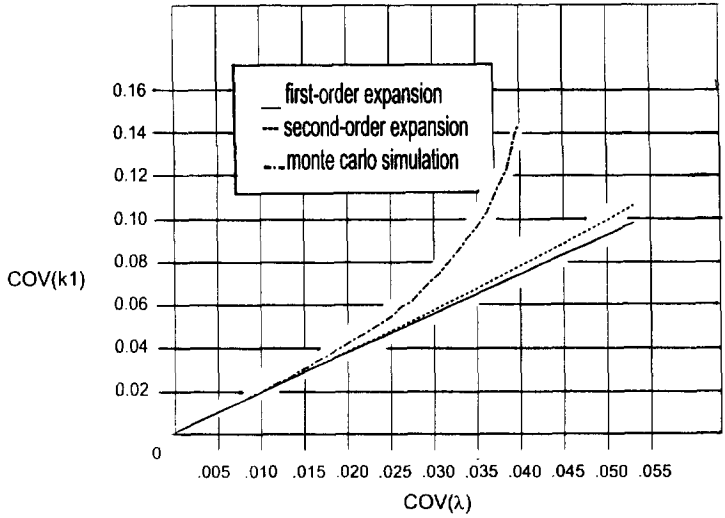

FIG. 4. Effect of COV_λ on COV_{k_1}, $k_3 = 10$.

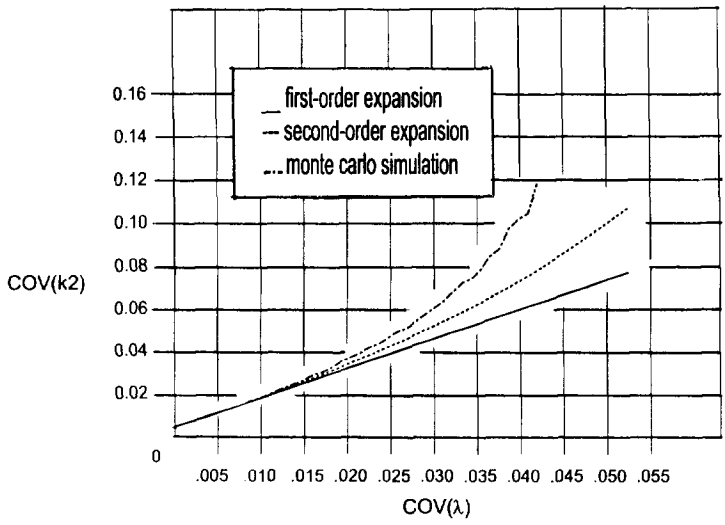

FIG. 5. Effect of COV_λ on COV_{k_2}, $k_3 = 10$.

TABLE 3
RESULTS FOR COV_λ OF 0.03, 2DOF SYSTEM WITH $k_3 = 100$

Method	Parameter	Results			Percent difference from Monte Carlo	
		Mean value	Standard deviation	Coefficient of variation	Mean value	Standard deviation
Monte Carlo simulation	m_1	10.018	0.31344	0.03129	—	—
	m_2	9.9817	0.31344	0.03140	—	—
	k_1	10.043	0.51232	0.05101	—	—
	k_2	9.9779	0.63429	0.06357	—	—
	k_3	99.818	4.9977	0.05007	—	—
First-order perturbation expansion	m_1	10.000	0.31316	0.03132	0.18	0.09
	m_2	10.000	0.31316	0.03132	0.18	0.09
	k_1	10.000	0.50120	0.05012	0.43	2.22
	k_2	10.000	0.62860	0.06286	0.22	0.91
	k_3	10.000	5.0284	0.05028	0.18	0.61
Second-order perturbation expansion	m_1	10.013	0.31362	0.03132	0.05	0.06
	m_2	9.9867	0.31362	0.03140	0.05	0.06
	k_1	10.047	0.50448	0.05021	0.04	1.55
	k_2	9.9664	0.62964	0.06318	0.12	0.74
	k_3	99.884	5.0324	0.05038	0.07	0.69

for the constrained system, while those of the unconstrained system obviously remain the same.

The results for this case are presented in Table 3. These results have been generated using $COV_\lambda = 0.03$, for comparison with previous results. It can be seen that the perturbation results are quite close to those of the Monte Carlo simulation. These results show much greater agreement than those in Table 2 for the same COV_λ.

Figures 6–8 show the variation of the system parameter coefficients of variation with increasing values of COV_λ. Note that the maximum value which can be used for Monte Carlo simulation is given by $COV_\lambda = 0.09321$ for this case, so that the range of these plots is greater than that of the previous example. The ranges on both axes are double those in Figures 3–5, so that their slopes may be directly compared.

These figures indicate that, with the exception of parameter k_1, the results are in good agreement across the entire range of the Monte Carlo simulation. Also, the slopes of these curves are, in general, smaller than those of Figures 3–5, indicating that the uncertainty in the eigenvalues creates smaller uncertainty in the output for this case. A physical explanation

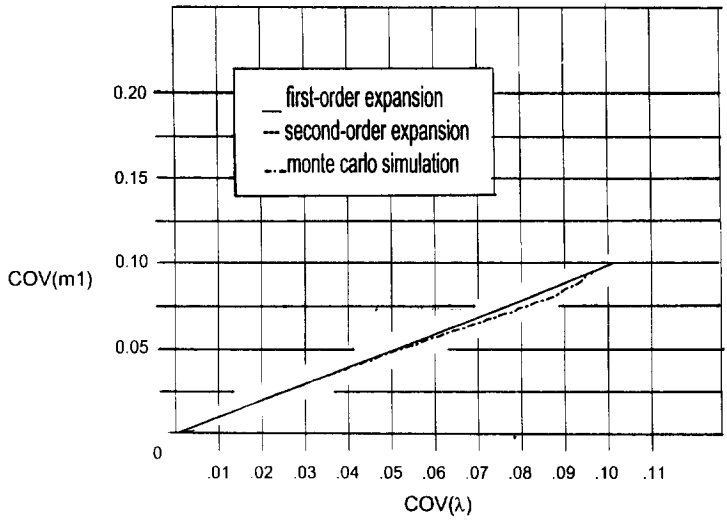

FIG. 6. Effect of COV_λ on COV_{m_1}, $k_3 = 100$.

of this would be that the stiffer boundary condition forces the two sets of eigenvalues farther apart (i.e., $\lambda_1^\bullet$ moves away from λ_1, and so on). In this way, uncertainties are smaller relative to the differences between the constrained and unconstrained eigenvalues. This system is thus somewhat less sensitive to eigenvalue uncertainty.

5. SENSITIVITY ANALYSIS

Thus far, the combined effects of all eigenvalue uncertainties have been examined. It is also of interest to study the effects of each individual eigenvalue on the calculated system parameters.

Figures 9 and 10 graphically show the (deterministic) system stiffnesses as functions of eigenvalues λ_1 and λ_2 for the first two-degree-of-freedom example given. These curves are created by varying one eigenvalue at a time, while holding the others constant. Again, the inequality

$$0 < \lambda_1 < \lambda_1^\bullet < \lambda_2 < \lambda_2^\bullet < \cdots < \lambda_N < \lambda_N^\bullet$$

is of importance since it determines the range over which each eigenvalue can be varied. For example, λ_1 can be varied only between 0 and $\lambda_1^\bullet$ (not including these endpoints), and so on.

The sensitivity of a parameter to uncertainty in an eigenvalue depends

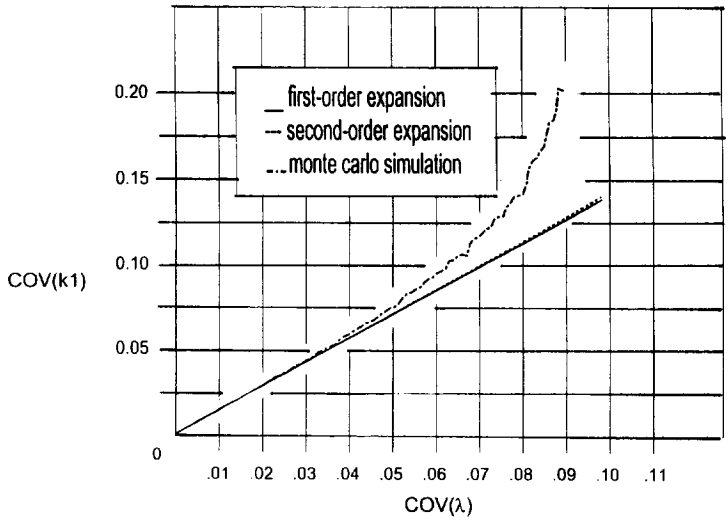

FIG. 7. Effect of COV_λ on COV_{k_1}, $k_3 = 100$.

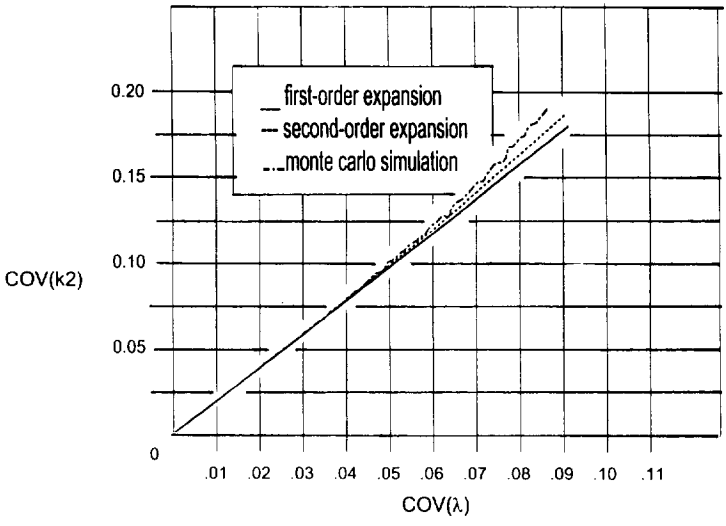

FIG. 8. Effect of COV_λ on COV_{k_2}, $k_3 = 100$.

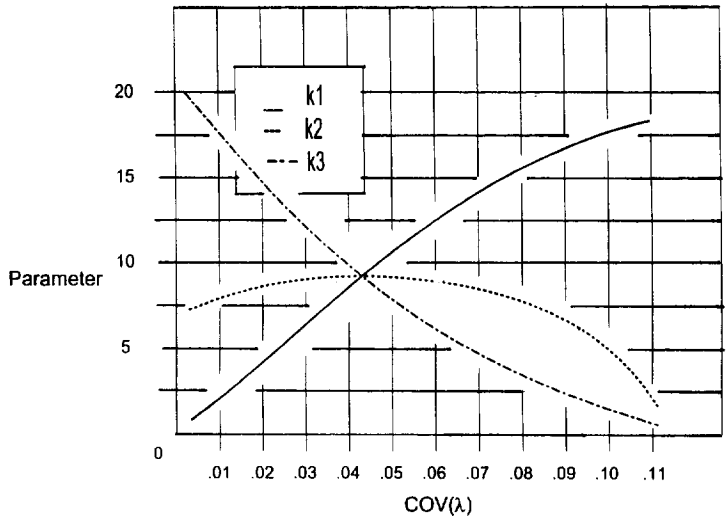

FIG. 9. Sensitivity of stiffnesses to λ_1.

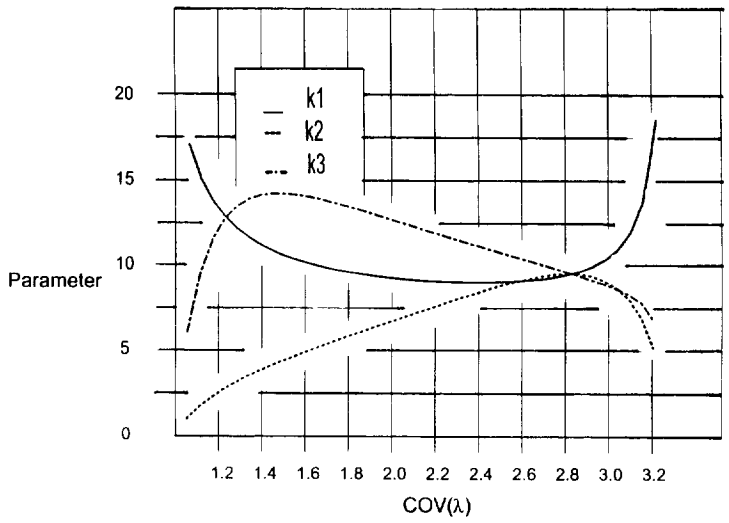

FIG. 10. Sensitivity of stiffnesses to λ_2.

on the slope and curvature of the appropriate curve at the point of interest. In this case, the springs and masses are all equal in value, so that this point is easily found where the curves meet.

The curves appear to be continuous and smooth over their respective ranges. It also appears that a linear or quadratic expression could approximate the curve quite well near the center of each range, but less well near the endpoints. These endpoints are of concern when the eigenvalues of the constrained system are close to those of the unconstrained system. Thus, if the constraint spring is chosen so that the eigenvalues are farther apart, the perturbation method should yield more accurate results. This is supported by the comparison of results given earlier.

6. CONCLUSIONS

A perturbation expansion has been used to derive expressions which estimate the statistics of a function of random variables. These have been applied to an inverse vibration problem. Thus, the means and standard deviations of the masses and spring constants of a discrete dynamic system can be determined from the statistics of the system's eigenvalue. The eigenvalues needed are those of the system of interest and those of the same system which has been constrained by an additional spring.

The first example considered was two-degree-of-freedom system. The constraint spring was of the same stiffness as those within the system. The results from all three methods can be seen to be very close to one another for small values of the eigenvalue coefficient of variation COV_λ. However, the results tend to diverge for larger COV_λ. The next example used the same basic system, but the stiffness of the constraint spring was an order of magnitude larger. The results here show good agreement over a larger range of values for COV_λ. Thus, the imposed boundary conditions have an important effect on the accuracy of the perturbation methods. In general, the uncertainty in eigenvalues result in smaller system parameter uncertainties when the stiffer spring is employed. Thus, the boundary conditions also affect the degree of uncertainty propagation in the mathematical model. In fact, it may be possible to choose a constraint stiffness so that the uncertainty in the output is minimized.

In examining the perturbation results, the Monte Carlo results are used as a basis for comparison, so that one may judge the "accuracy" of these results. In nearly all cases, the second-order approximation is found to be more accurate than the first-order approximation. However, in many of these situations the difference is negligible. In most cases where this difference is appreciable, the second-order estimate still shows a great deal

of error. These factors indicate that any added accuracy of the second-order approximation may not be great enough to justify the computational effort required in its calculation. The first-order expansion may also be more useful since it requires fewer assumptions on the nature of the eigenvalue probability density functions.

This work was carried out in partial fulfillment of an M.S. degree in the Department of Mechanical and Aerospace Engineering, Rutgers University. The first author wishes to thank the department for their support.

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